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Realization of Modified Cut-Set Matrix and Applications

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Abstract—The concepts of network of departure and padding n-port network are defined and certain important properties of these networks are stated. A necessary and sufficient criterion for the realization of the modified cut-set matrix of a resistive n-port network having a prescribed configuration is given. A new approach to the synthesis of short-circuit conductance matrix of a resistive n-port network with more than $(n+1)$ nodes is outlined. Necessary and sufficient conditions for the realization of the potential factor matrix of an $(n+2)$ node resistive n-port network are also obtained. These conditions lead to a simple test for the realizability of the modified cut-set matrix of an $(n+2)$ node n-port network. Examples are worked out to illustrate the application of these results.

I. Introduction

In this paper we investigate the problem of realization of the modified cut-set matrix [1] of a resistive n-port network and its relation to the realization of the short-circuit conductance and the potential factor matrices of such networks. We first introduce the notation to be followed.

We consider a resistive n-port network N, having no internal vertices. The linear graph G^* of N is assumed to be complete and edges with zero conductances permitted. Let T, the port configuration of N, be in r connected parts $T_1, T_2, \cdots, T_r$. The edges of T, called the port edges, are oriented according to the orientations of the ports. Let T be a subgraph of a tree T_0 of G^*. The edges of T_0 other than those of T are termed nonport edges and may be oriented arbitrarily. Let

$$C_0 = \left[\frac{C_1}{C_2}\right]$$

be the fundamental cut-set matrix of G^* with respect to T_0, where the rows of C_1 correspond to the port edges and those of C_2 correspond to the nonport edges. If G is the diagonal matrix of edge conductances of N, then the cut-set admittance matrix Y_0 of N is given by

$$Y_0 = \left[\begin{array}{c|c} C_1GC_1' & C_1GC_2' \\ \hline C_2GC_1' & C_2GC_2' \end{array}\right] = \left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right] \tag{1}$$

in which the rows and columns of Y_{11} correspond to the port edges and those of Y_{22} correspond to the nonport edges.

The modified cut-set matrix C of N is defined as [1]

$$C = C_1 - Y_{12}Y_{22}^{-1}C_2. \tag{2}$$

It can be shown that after rearranging the rows and the columns of C, the partition will be as follows.

$$C = [C^P \mid C^N] \tag{3}$$

where

$$C^P = \begin{bmatrix} C^1 & & & 0 \\ & C^2 & & \\ 0 & & \ddots & \\ & & & C^r \end{bmatrix}$$

with C^i being the fundamental cut-set matrix with respect to T_i of the complete graph built on the vertices of T_i.

The columns of C^P correspond to edges joining a pair of vertices in the same connected part of T, while the columns of C^N correspond to edges joining a pair of vertices in two different connected parts of T. For an n-port network containing no negative conductances the magnitude of each entry of C^N is less than unity.

The potential factor matrix [2], [3] $K = [k_{ij}]$ of the n-port network N is defined as the $n \times n$ matrix, where k_{ij}, called the potential factor of port j with respect to port i, is the potential of the positive reference terminal of port j with respect to the negative reference terminal of port i when port i is excited with a source of unit voltage

Manuscript received June 30, 1969; revised April 2, 1970.
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and all the other ports are short-circuited. After rearranging its rows and columns, the matrix K can be partitioned as follows.

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1r} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2r} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{r1} & K_{r2} & K_{r3} & \cdots & K_{rr} \end{bmatrix} \quad (4)$$

where

1) each submatrix K_{ii}, $i = 1, 2, \cdots, r$ has entries comprising 1 and 0 only and is uniquely fixed by the configuration of T_i;
2) each entry in a submatrix K_{ij}, $j \neq i$ is less than unity and greater than zero, except in degenerate cases;
3) all entries in any row of each K_{ij}, $j \neq i$ are equal.

II. Network of Departure, Padding n-Port Network, and Realization of the Modified Cut-Set Matrix

In this section we first introduce the concepts of the network of departure and the padding n-port network and outline their important properties. We then obtain a necessary and sufficient condition for the realization of the modified cut-set matrix of a resistive n-port network having a prescribed port configuration.

A. Network of Departure and Padding n-Port Network

Theorem 1:

a) Let N represent an n-port network having a port configuration T. Let the cut-set admittance matrix Y_0 of N with respect to a tree T_0 of N, of which T is a subgraph be given by

$$Y_0 = \begin{array}{c} \\ \updownarrow n \\ \\ \end{array} \overset{\leftarrow n \rightarrow}{\left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right]},$$

where the rows of the $n \times n$ matrix Y correspond to the edges of T, i.e., the port edges. Then the cut-set admittance matrix Y_0^* of N with respect to any other tree T_0^* of which T is a subgraph is also equal to Y_0.

b) Let N and N^* be two $(n + k)$ node realizations, $k > 1$, of the same n-port short-circuit conductance matrix Y with a prescribed port configuration T. Let T_0 and T_0^* be two distinct trees such that T is a subgraph of both. If the cut-set admittance matrix of N with respect to T_0 and the corresponding matrix of N^* with respect to the tree T_0^* are both equal to

$$\begin{array}{c} \updownarrow n \\ \updownarrow k-1 \end{array} \overset{\leftarrow n \rightarrow}{\left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right]};$$

where the rows of Y correspond to the edges of T (port edges) in each case, then the two networks N and N^* are identical.

Proof:

a) Let C_0 and C_0^* refer respectively to the fundamental cut-set matrices of N with respect to T_0 and T_0^*, with the first n rows of C_0 and C_0^* corresponding to the edges of T. Then C_0^* can be expressed as $C_0^* = AC_0$, where the transformation matrix A is of the form

$$A = \left[\begin{array}{c|c} U & A_1 \\ \hline 0 & A_2 \end{array}\right] \begin{array}{c} \updownarrow n \\ \\ \end{array}.$$

We have

$$Y_0 = \left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right] = C_0GC_0',$$

where G is the edge conductance matrix of N.

Now,

$$Y_0^* = C_0^*G(C_0^*)' = AC_0GC_0'A' = AY_0A' = Y_0.$$

Hence Theorem 1a).

b) It follows from Theorem 1a) that the cut-set admittance matrix of N^* with respect to T_0 is also equal to

$$\left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right]$$

where the first n rows correspond to the edges of T.

Since the two networks N and N^* have the same cut-set admittance matrix with respect to a common tree configuration T_0, they must be identical.

Definition 1—Network of Departure: An n-port network with more than $(n + 1)$ nodes is called a network of departure N_d with respect to a real symmetric matrix Y if its cut-set admittance matrix is equal to

$$\left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right]$$

where the rows of Y correspond to the port edges.

We note that the n-port short-circuit conductance matrix of N_d is equal to Y. It follows from Theorem 1 that, given the port configuration and the short-circuit conductance matrix, the corresponding network of departure N_d is unique. Thus for a given n-port network N, there exists a network of departure that is uniquely determined and referred to as the network of departure of the given n-port network N.

Definition 2—Padding n-Port Network: An n-port network is called a padding n-port network N_p if its short-circuit conductance matrix is equal to zero.

Ignoring the degenerate case of a network having zero conductance for all its edges, we note that N_p should necessarily contain both positive and negative conductances. We further note that the short-circuit conductance matrix of the parallel combination of N_p and N_d, both having identical port configurations and orientations, is equal to that of N_d. We also note that any n-port network N can be considered as the parallel combination of a network of departure N_d and a padding n-port network N_p [4]. Since for a given N, N_d is uniquely determined, the corresponding N_p is also uniquely fixed.

Let the set of vertices in the ith connected part T_i of the port configuration T of N be denoted as $i_1, i_2, \cdots, i_{n_i}$. Let $(g_{i_k j_m})$, $(g_{i_k j_m})_p$, and $(g_{i_k j_m})_d$ refer to the conductances of edges connecting the vertices i_k and j_m in the network N, the associated padding network N_p, and the network of departure N_d, respectively.

Now,

$$g_{i_k j_m} = (g_{i_k j_m})_p + (g_{i_k j_m})_d. \tag{5}$$

Let

$$S_{i_k j} = \sum_{m=1}^{n_j} g_{i_k j_m}, \tag{6}$$

$$(S_{i_k j})_d = \sum_{m=1}^{n_j} (g_{i_k j_m})_d, \tag{7}$$

$$(S_{i_k j})_p = \sum_{m=1}^{n_j} (g_{i_k j_m})_p. \tag{8}$$

Each of the above quantities is thus equal to the sum of the conductances of all the edges joining the kth vertex in T_i to the vertices of T_j in the respective network. Then it follows that

$$S_{i_k j} = (S_{i_k j})_d + (S_{i_k j})_p. \tag{9}$$

With this notation we next proceed to list the important properties of N_d and N_p. The proofs of all the properties, except the last two, are not given here but may be readily constructed.

Properties of N_d:

a) The modified cut-set matrix of N_d does not exist.

b) The n-port short-circuit conductance matrix of the parallel combination of N_d with any other n-port network N_0 having a determinate short-circuit conductance matrix and the same port configuration and orientation as N_d is equal to the sum of the short-circuit conductance matrices of N_d and N_0.

This property implies that the parallel combination of N_d and N_0 is always proper, for any N_0 satisfying the stated restrictions.

c) A necessary and sufficient condition that an n-port network N_d be a network of departure is that

$$(S_{i_k j})_d = 0 \qquad \text{for all } i, j = 1, 2, \cdots, r, j \neq i, \quad \text{and } k = 1, 2, \cdots, n_i. \tag{10}$$

d) If an n-port network N contains no negative conductances, then in the corresponding network of departure N_d, the conductance of every edge joining the vertices in the same connected part of T is nonnegative.

Properties of N_p:

e) For a given N, the modified cut-set matrix of the corresponding padding network N_p is the same as that of N.

f) An $(n + 1)$ node-padding n-port network is the degenerate network with all its edge conductances equal to zero.

g) If N contains no negative conductances, then in the corresponding padding network N_p,

$$(S_{i_k j})_p \geq 0 \qquad \text{for all } i \text{ and } j = 1, 2, \cdots, r, j \neq i \quad \text{and } k = 1, 2, \cdots, n_i.$$

h) For a padding network N_p with $(S_{i_k j})_p \geq 0$ for all i and j, $j \neq i$ and $k = 1, 2, \cdots, n_i$, the following relation must be satisfied.

$$(g_{i_k i_m})_p \leq 0 \qquad \text{for } k, m = 1, 2, \cdots, n_i, k \neq m, \quad \text{and } i = 1, 2, \cdots, r.$$

Proof: Consider N_p with all ports except those in T_i short-circuited. Each subtree T_j, $j \neq i$ is now reduced to a single vertex j. The conductance of a typical edge in this new network is either $(g_{i_k i_m})_p$ or $(S_{i_k j})_p$. If we eliminate by generalized star-mesh conversion the edges of the second type, there results a network in which the edges joining vertices i_k and i_m have a conductance $(g_{i_k i_m})_p + (g_{i_k i_m})_c$ where the latter quantity is nonnegative. Now the resulting $(n_i - 1)$-port network constructed on n_i vertices has a short-circuit conductance matrix identically equal to zero. Hence by virtue of Property f), we have

$$(g_{i_k i_m})_p + (g_{i_k i_m})_c = 0.$$

Since $(g_{i_k i_m})_c$ is nonnegative, we have from the above, $(g_{i_k i_m})_p \leq 0$.

i) Given a padding n-port network N_p such that

$$(S_{i_k j})_p \geq 0 \qquad \text{for all } i \text{ and } j = 1, 2, \cdots r, j \neq i, \quad \text{and } k = 1, 2, \cdots, n_i$$

then a network of departure N_d can always be obtained such that the parallel combination of N_p and N_d contains no negative conductances.

Proof: First we note that for the given padding n-port network N_p,

$$(g_{i_k i_m})_p \leq 0 \quad \text{for all } i = 1, 2, \cdots, r, \quad \text{and } k, m = 1, 2, \cdots, n_i, k \neq m$$

by virtue of Property h). We proceed to prove Property i) by giving a procedure for constructing a suitable N_d.

Construct a network N_x having the same port configuration and orientations as N_p and such that

$$(g_{i_k i_m})_x = -(g_{i_k i_m})_p \quad \text{for all } k, m = 1, 2, \cdots, n_i, \quad k \neq m, \text{ and } i = 1, 2, \cdots, r \tag{11}$$

$$(g_{i_k j_m})_x = -(g_{i_k j_m})_p + \frac{(S_{i_k j})_p \times (S_{j_m i})_p}{\Delta_{ij}}$$
$$\text{for all } i, j = 1, 2, \cdots, r, \quad j \neq i, \quad \text{and } k = 1, 2, \cdots, n_i, \quad m = 1, 2, \cdots, n_j \tag{12}$$

where

$$\Delta_{ij} = \sum_{k=1}^{n_i} (S_{i_k j})_p = \sum_{m=1}^{n_j} (S_{j_m i})_p.$$

For the network N_x constructed as above,

$$\begin{aligned}(S_{i_k j})_x &= \sum_{m=1}^{n_j} (g_{i_k j_m})_x \\ &= -\sum_{m=1}^{n_j} (g_{i_k j_m})_p + \frac{(S_{i_k j})_p \times \sum_{m=1}^{n_j} (S_{j_m i})_p}{\Delta_{ij}} \\ &= -\sum_{m=1}^{n_j} (g_{i_k j_m})_p + (S_{i_k j})_p \\ &= 0, \quad \text{for all } i \text{ and } j = 1, 2, \cdots, r, \quad j \neq i, \quad \text{and } k = 1, 2, \cdots, n_i. \end{aligned} \tag{13}$$

It follows from Property c) and (13) that the network N_x constructed with its conductances satisfying (11) and (12) is a network of departure N_d.

The conductance $g_{i_k j_m}$ in the parallel combination of N_d and N_p is given by

$$\begin{aligned} g_{i_k j_m} &= (g_{i_k j_m})_p + (g_{i_k j_m})_d \\ &= (g_{i_k j_m})_p - (g_{i_k j_m})_p + \frac{(S_{i_k j})_p \times (S_{j_m i})_p}{\Delta_{ij}} \\ &= \frac{(S_{i_k j})_p \times (S_{j_m i})_p}{\Delta_{ij}} \geq 0 \\ &\text{for all } i, j = 1, 2, \cdots, r, \quad j \neq i \\ &k = 1, 2, \cdots, n_i \\ &\text{and } m = 1, 2, \cdots, n_j. \end{aligned} \tag{14}$$

Also

$$\begin{aligned} g_{i_k i_m} &= (g_{i_k i_m})_p + (g_{i_k i_m})_d \\ &= 0 \quad \text{for all } i = 1, 2, \cdots, r, \\ &k, m = 1, 2, \cdots, n_i, \quad k \neq m. \end{aligned} \tag{15}$$

From the above equations it follows that the parallel combination of the given N_p and the network of departure N_d constructed to satisfy (11) and (12) contain no negative conductances.

B. Realization of the Modified Cut-Set Matrix of an n-Port Network

Let there be given a real matrix C partitionable as in (3) and such that C^i, $i = 1, 2, \cdots, r$ is realizable as the fundamental cut-set matrix with respect to T_i, $i = 1, 2, \cdots, r$. Consider an n-port network N having a port configuration defined by T_i, $i = 1, 2, \cdots, r$. The graph of N is assumed to be complete. Each edge of N corresponds to a unique column of C and this correspondence is assumed to be known. It is required to obtain a necessary and sufficient condition that C represent the modified cut-set matrix of an N containing no negative conductances.

In the light of Properties e)–i), the above problem can be considered equivalent to obtaining necessary and sufficient conditions for the realization of C by a padding n-port network N_p whose conductances satisfy the property stated in Property i).

Thus, we have the following theorem on the realization of the modified cut-set matrix C.

Theorem 2: If and only if there exists a real diagonal matrix G_p such that

$$CG_pC_1' = 0$$
$$CG_pC_2' = 0$$
$$\det [C_2G_pC_2'] \neq 0 \quad \text{and}$$
$$(S_{i_k j})_p \geq 0 \quad \text{for all } i, j = 1, 2, \cdots, r, \quad j \neq i, \quad \text{and } k = 1, 2, \cdots, n_i,$$

the matrix C can be realized as the modified cut-set matrix of an n-port network containing no negative conductances.

Proof—Necessity: This follows from Properties e)–i) and Theorem 4 of [3].

Proof—Sufficiency: This follows from Properties e)–i) and Theorems 4 and 6 of [3].

The usefulness of the results of this section in the synthesis of the short-circuit conductance matrices of resistive n-port networks will be considered in the next section.

III. Synthesis of the Short-Circuit Conductance Matrix of a Resistive n-Port Network

The problem of realization of the short-circuit conductance matrix Y of a resistive n-port network has received the attention of research workers in network theory for more than a decade. Whereas the problem of synthesis of n-ports with $(n + 1)$ nodes is considered solved, the problem of synthesis with more than $(n + 1)$ nodes is yet unsolved. Extending Cederbaum's approach for $(n + 1)$ node network synthesis, one approach to the above problem may be to decompose the given matrix Y as $Y = CGC'$ where C is a real matrix and G is a real diagonal matrix of nonnegative numbers, and then to realize C as the modified cut-set matrix of a resistive n-port network. The successful application of this procedure, however, requires as a first step, a procedure for the decomposition of Y, which is not known. Further, it is known that such a decomposition is not unique.

A slightly less difficult problem is to consider the realization of a real symmetric matrix as the short-circuit conductance matrix of an n-port network having a specified port configuration. The earliest approach suggested by Guillemin [4] to solve this problem essentially requires the determination of a) the unique network of departure N_d with respect to Y having the prescribed port configuration, and b) a suitable padding n-port network N_p so that the parallel combination of N_p and N_d contains no negative conductances. The procedure given by Guillemin to generate padding n-port networks is general, in that it can generate all possible padding n-port networks. However, in the light of the discussions of the last section, it follows that only the class of padding n-port networks whose conductances satisfy the property stated in Property i) are required. Hence, while applying Guillemin's procedure this constraint on conductances should be incorporated.

Other approaches due to Frisch and Swaminathan [5] and Halkias and Lupo [6] can also be stated in terms of networks of departure and padding networks. They differ from that of Guillemin only in the method used to generate padding networks. In generating a padding network, Frisch and Swaminathan first obtain the cut-set admittance of the padding network in terms of $n(r-1)+[r(r-1)]/2$ arbitrarily assumed numbers and then obtain the edge conductance values from the cut-set admittance matrix. A significant result of this investigation is the derivation of the supremacy condition that is necessary for the realizability of the Y matrix of $(n+2)$ node-resistive n-port networks. This condition is simple and easy to apply. Halikias and Lupo have obtained formulas that express directly the edge conductances of $(n+2)$ node-padding n-port networks in terms of $(n+2)$ arbitrarily assumed numbers. The extension of this approach to generate padding networks with more than $(n+2)$ nodes, given later by Lupo [7], is, however, not general, since it generates only a class of padding networks whose potential factors are related in a special manner.

In this section we present an alternate procedure for Y-matrix synthesis, which again differs from that of Guillemin only in the method used to generate padding networks. The procedure is as follows.

Step 1: For the given Y matrix obtain the unique network of departure N_d having the prescribed port configuration.

Step 2: Assuming suitable values for nonzero nonunity potential factors and using Theorem 2 of [2], construct the modified cut-set matrix C appropriate to the given port configuration. Obtain a padding n-port network N_p such that $(S_{i_kj})_p \geq 0$ for all i and j, $j \neq i$, $i, j = 1, 2, \cdots r$ and $k = 1, 2, \cdots, n_i$. For this the procedure contained in Theorem 2 may be used.

Step 3: If, for one assumed C, no N_p can be obtained such that the parallel combination N of N_p and N_d contains no negative conductances, assume a different set of values for the nonzero nonunity potential factors and hence a different modified cut-set matrix. Repeat Step 2.

Step 4: If an N_p can be found, which results in an N containing no negative conductances, then this network N will be a proper realization of Y. If not, the matrix Y is not realizable by resistive n-port networks having the prescribed port configuration.

A significant feature of the method used in Step 2 of the above procedure to generate padding networks is that edge conductances of padding networks are expressed in terms of potential factors. This will be helpful in extending the results of resistive n-port synthesis to the synthesis of Y matrices of RLC networks, since, in the synthesis of the latter, all residue matrices, if they are real, or all parameter matrices are required to be realized by networks having identical potential factors [8].

This new method to generate padding networks requires the assumption of suitable values for nonzero nonunity potential factors. Since these potential factors should be the same as those for the final network N, to avoid excessive computational labor, it is necessary that we obtain necessary and sufficient conditions, which the nonzero nonunity potential factors of resistive n-port networks containing no negative conductances should satisfy. This leads us to the problem of synthesis of the potential factor matrix of resistive n-port networks. In the next section we consider this aspect of the Y-matrix synthesis problem in relation to an $(n+2)$ node network.

IV. Realization of the Potential Factor Matrix of an $(n+2)$ Node-Resistive n-Port Network

In this section we obtain a necessary and sufficient condition which the elements of K_{12} and K_{21} should satisfy in order that

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

be realizable as the potential factor matrix of an $(n+2)$ node-resistive n-port network whose port configuration defines K_{11} and K_{22}. It can be shown that such a condition provides an effective criterion for the realization of the modified cut-set matrix of $(n+2)$ node-resistive n-port networks. We also illustrate the application of this condition to the synthesis of the short-circuit conductance matrix of $(n+2)$ node n-port networks.

We first consider a resistive $(n+2)$ node n-port network N containing no negative conductances. We assume, without loss of generality as will be shown later, that T_1 and T_2 of N are linear trees. In designating the vertices, edges, etc., of N, we adopt a notation different from the one used earlier and simpler in the present context. Let the vertices of any linear tree T_0 of N, of which T_1 and T_2 are subgraphs, be numbered consecutively starting from one end vertex of T_0. Let there be m edges in T_1 and $(n-m)$ edges in T_2. Let e_{ij} represent the edge-connecting vertices i and j. Let $e_{i,i+1}$ denote port i if $i \leq m$ and $e_{i+1,i+2}$ denote port i if $i \geq m+1$. $e_{m+1,m+2}$ will then

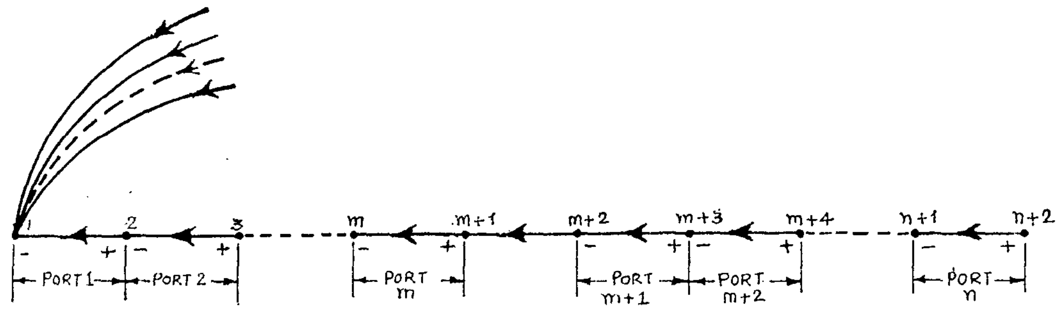

Fig. 1. Notation followed in vertex designation, port and edge orientation.

represent the nonport edge of T_0 as shown in Fig. 1. Let the orientation of e_{ij} be away from j, if $j > i$, and let g_{ij} refer to the conductance of edge e_{ij}.

Let

$$S_{i,2} = \sum_{j=m+2}^{n+2} g_{ij}, \qquad i \leq m+1$$
$$S_{i,1} = \sum_{j=1}^{m+1} g_{ij}, \qquad i \geq m+2. \tag{16}$$

Let

$$S = \sum_{i=1}^{m+1} S_{i,2} = \sum_{i=m+2}^{n+2} S_{i,1}. \tag{17}$$

Let the modified cut-set matrix C and the submatrix C_1 of the fundamental cut-set matrix C_0 be partitioned as

$$C = \begin{bmatrix} C_a \\ C_b \end{bmatrix}; \qquad C_1 = \begin{bmatrix} C_{1a} \\ C_{1b} \end{bmatrix} \tag{18}$$

where the rows of C_a and C_{1a} correspond to the port edges in T_1 and those of C_b and C_{1b} correspond to the port edges in T_2.

Let the columns of C_a be rearranged and partitioned as

$$C_a = [C_{a,1} \mid C_{a,2} \mid \cdots \mid C_{a,i} \mid \cdots \mid C_{a,m+1} \mid C_{a,T_1} \mid C_{a,T_2}] \tag{19}$$

where $C_{a,i}$ consists of all columns of C_a corresponding to the edges joining vertex i in T_1 to all vertices in T_2; C_{a,T_i} consists of all columns of C_a corresponding to the edges connecting two vertices in T_i.

Similarly the columns of C_b, C_{1b}, C_{1a}, and C_2 can be arranged and partitioned. We note that

$$\begin{aligned} C_{b,T_1} &= 0 \qquad & C_{a,T_2} &= 0 \\ C_{1a,T_2} &= 0 \qquad & C_{1b,T_1} &= 0 \\ C_{2,T_1} &= 0 \qquad & C_{2,T_2} &= 0 \end{aligned} \tag{20}$$

and that all the entries in each of the row matrices $C_{2,i}$, $i = 1, 2, \cdots, m+1$ are equal to 1.

Let g_i be the column matrix of conductances of edges corresponding to the columns of $C_{a,i}$, $C_{b,i}$, etc., with its rows arranged in the same order as the columns of $C_{a,i}$, $C_{b,i}$, etc.

We have, as already mentioned in Section I, that

$$k_{ij} = k_{ik} \qquad i \leq m \quad \text{and} \quad j, k \geq m+1$$
$$\text{or} \quad i \geq m+1 \quad \text{and} \quad j, k \leq m. \tag{21}$$

Let

$$k_i = k_{i,m+1} = k_{i,m+2} = \cdots = k_{i,n} \qquad i \leq m$$
and
$$k_i = k_{i,1} = k_{i,2} = \cdots = k_{i,m} \qquad i \geq m+1. \tag{22}$$

It follows from Theorem 4 of [3] that the diagonal matrix G of edge conductances of N satisfies

$$CGC_2' = 0$$

i.e.,

$$C_aGC_2' = 0 \tag{23a}$$

and

$$C_bGC_2' = 0. \tag{23b}$$

It can be shown from (23a) that

$$k_i = \frac{\sum_{j=i+1}^{m+1} S_{j,2}}{S} \qquad i = 1, 2, \cdots, m. \tag{24}$$

Thus, for a network of nonnegative conductances, we have from (24)

$$k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_m. \tag{25}$$

Also it can be shown from (23b) that

$$k_i = \frac{\sum_{j=i+2}^{n+2} S_{j,1}}{S} \qquad i = m+1, m+2, \cdots, n \tag{26}$$

and it follows that

$$k_{m+1} \geq k_{m+2} \geq k_{m+3} \geq \cdots \geq k_n. \tag{27}$$

Hence the elements of the submatrices K_{12} and K_{21} of the potential factor matrix K of an $(n+2)$ node-resistive n-port network having a linear 2-tree port configuration and no negative conductances must satisfy (21), (25), and (27). We note that K_{11} and K_{22} for such a network will have the following form.

$$K_{11} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} m \text{ rows}$$

$$K_{22} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} n - m \text{ rows.} \qquad (28)$$

Next we show that if the elements of the submatrices K_{12} and K_{21} satisfy (21), (25), and (27) and if K_{11} and K_{22} are of the forms specified in (28), then

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

represents the potential factor matrix of an $(n + 2)$ node-resistive n-port network containing no negative conductances and having a linear 2-tree port configuration T defined by K_{11} and K_{22}.

Using K, we first construct the modified cut-set matrix C appropriate to the port configuration T. This can always be done in view of Theorem 2 of [2]. We then show that it is possible to construct a padding n-port network N having the port configuration T and the modified cut-set matrix C and satisfying the constraint

$$S_{i,2} \geq 0 \qquad \text{for all} \quad i \leq m + 1$$

$$S_{i,1} \geq 0 \qquad \text{for all} \quad i \geq m + 2.$$

For this we adopt the procedure contained in Theorem 2.

Let G be the diagonal matrix of edge conductances of the required padding n-port network. We seek a solution for G of the following equations.

$$CGC_1' = 0 \qquad (29a)$$

$$CGC_2' = 0 \qquad (29b)$$

$$\det [C_2GC_2'] \neq 0 \qquad (29c)$$

$$S_{i,2} \geq 0 \qquad \text{for all} \quad i \leq m + 1 \qquad (29d)$$

$$S_{i,1} \geq 0 \qquad \text{for all} \quad i \geq m + 2. \qquad (29e)$$

We note that $\det [C_2GC_2'] \neq 0$ if (29d) and (29e) are satisfied and if further $S_{i,2} > 0$ for some $i \leq m + 1$. We also note that if $C_aGC_{1b}' = 0$ then $C_bGC_{1a}' = 0$ since CGC_1' is symmetric. Thus, (29a–e) are equivalent to the following.

$$C_aGC_2' = 0 \qquad (30a)$$

$$C_bGC_2' = 0 \qquad (30b)$$

$$C_aGC_{1a}' = 0 \qquad (30c)$$

$$C_bGC_{1b}' = 0 \qquad (30d)$$

$$C_bGC_{1a}' = 0 \qquad (30e)$$

$$S_{i,2} \geq 0 \qquad \text{for all} \qquad i \leq m + 1 \qquad (30f)$$

$$S_{i,1} \geq 0 \qquad \text{for all} \qquad i \geq m + 2 \qquad (30g)$$

$$S_{i,2} > 0 \qquad \text{for some} \quad i \leq m + 1. \qquad (30h)$$

From (24), which can be obtained from (30a), we get the following

$$\begin{aligned} S_{i,2} &= (k_{i-1} - k_i)S, \qquad i = 2, 3, \cdots, m \\ S_{1,2} &= (1 - k_1)S \\ S_{m+1,2} &= k_mS. \end{aligned} \qquad (31)$$

Next consider (30b) and (30e). After setting $C_{b,T_1} = 0$, $C_{2,T_1} = 0$, and $C_{2,T_2} = 0$, (30b) can be written as

$$C_{b,1}\{g_1\} + \cdots + C_{b,i}\{g_i\} + \cdots + C_{b,m+1}\{g_{m+1}\} = 0. \qquad (32)$$

We obtain from (30e) the following sets of equations

$$C_{b,1}\{g_1\} + \cdots + C_{b,i}\{g_i\} = 0 \qquad i = 1, 2, \cdots, m. \qquad (33)$$

Simultaneous solution of (32) and (33) yields the following.

$$C_{b,i}\{g_i\} = 0 \qquad i = 1, 2, \cdots, m + 1. \qquad (34)$$

We note that $C_{b,1} = C_{b,2} = \cdots = C_{b,m+1}$. After expressing $C_{b,i}$ in terms of potential factors, (34) becomes

$$\begin{array}{ccccc} e_{i,m+2} & e_{i,m+3} & e_{i,m+4} & \cdots & e_{i,n+2} \end{array}$$
$$\begin{bmatrix} -k_{m+1} & 1 - k_{m+1} & 1 - k_{m+1} & \cdots & 1 - k_{m+1} \\ -k_{m+2} & -k_{m+2} & 1 - k_{m+2} & \cdots & 1 - k_{m+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_n & -k_n & -k_n & \cdots & 1 - k_n \end{bmatrix} \begin{bmatrix} g_{i,m+2} \\ g_{i,m+3} \\ \vdots \\ g_{i,n+2} \end{bmatrix} = 0 \qquad i = 1, 2, \cdots, m + 1. \qquad (35)$$

We obtain from the above

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} g_{i,m+2} \\ g_{i,m+3} \\ \vdots \\ g_{i,n+2} \end{bmatrix} = S_{i,2} \begin{bmatrix} k_{m+1} \\ k_{m+2} \\ \vdots \\ k_n \end{bmatrix},$$

$$i = 1, 2, \cdots, m+1. \tag{36}$$

Solving (36) we get,

$$g_{ij} = (k_{j-2} - k_{j-1}) S_{i,2} \qquad i = 1, 2, \cdots, m+1$$
$$j = m+3, \cdots, n+1$$

and

$$\begin{aligned} g_{i,m+2} &= (1 - k_{m+1}) S_{i,2} \qquad i = 1, 2, \cdots, m+1 \\ g_{i,n+2} &= k_n S_{i,2} \qquad i = 1, 2, \cdots, m+1. \end{aligned} \tag{37}$$

Consider (31) and (37). Choosing any positive value for S, the values obtained for $S_{i,2}$ from (31) will be nonnegative, since $k_1 \geq k_2 \geq \cdots \geq k_m$. Further, at least two of them will be positive. Hence, such a choice of $S_{i,2}$ will satisfy (30f), (30g), and (30h). We then use these values for $S_{i,2}$ and (37) to calculate the values of conductances of edges connecting vertices in T_1 to vertices in T_2. We note that the conductances of these edges will be nonnegative since $1 \geq k_{m+1} \geq k_{m+2} \geq \cdots \geq k_n$.

The following expression for conductances of edges connecting vertices in T_1 to vertices in T_2 can also be obtained using (26) and (37).

$$g_{ij} = \frac{S_{i,2} S_{j,1}}{S} \qquad \text{for all } i \text{ in } T_1 \text{ and } j \text{ in } T_2. \tag{38}$$

It only remains to obtain the conductances of edges connecting the vertices in T_1 only and also of the edges connecting vertices in T_2 only.

$$g_{ij} = \frac{-S_{i,2} S_{j,2}}{S} \qquad i = 1, 2, \cdots, m+1$$
$$j = 1, 2, \cdots, m+1, \quad j \neq i$$

$$g_{ij} = \frac{-S_{i,1} S_{j,1}}{S}$$
$$= -(k_{i-2} - k_{i-1})(k_{j-2} - k_{j-1}) S,$$
$$i, j = m+3, m+4, \cdots, n+1, \quad j \neq i.$$

$$g_{m+2,j} = \frac{-S_{m+2,1} S_{j,1}}{S}$$
$$= -(1 - k_{m+1})(k_{j-2} - k_{j-1}) S$$
$$j = m+3, \cdots, n+1$$

$$g_{i,n+2} = \frac{-S_{i,1} S_{n+2,1}}{S}$$
$$= -(k_{i-2} - k_{i-1}) k_n S \qquad i = m+3, \cdots, n+1$$

and

$$g_{m+2,n+2} = \frac{-S_{m+2,1} S_{n+2,1}}{S}$$
$$= -(1 - k_{m+1}) k_n S. \tag{39}$$

Summarizing the results of the above discussions, we outline below the steps to be followed in obtaining a padding n-port network N having a linear 2-tree port configuration and the modified cut-set matrix C and satisfying the constraint

$$S_{i,2} \geq 0 \qquad \text{for all } i \leq m+1$$
$$S_{i,1} \geq 0 \qquad \text{for all } i \geq m+2$$

and

$$S_{i,2} > 0 \qquad \text{for some } i \leq m+1.$$

Step 1: Choose any positive value for S. Obtain using (31) all

$$S_{i,2} \qquad i = 1, 2, \cdots, m+1.$$

Step 2: Obtain the conductances of the edges connecting vertices in T_1 to those in T_2 using (37).

Step 3: The conductances of edges connecting any two vertices in T_1 and of edges connecting any two vertices in T_2 can be obtained using (39).

It follows from the above and Theorem 2 that an $(n+2)$ node-resistive n-port network containing no negative conductances, having a linear 2-tree port configuration defined by K_{11} and K_{22}, and having the modified cut-set matrix C can always be constructed. Since matrix C was obtained from

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

it follows from Theorem 9 of [3] that such a network will have K as its potential factor matrix, when K_{12} and K_{21} satisfy (21), (25), and (27). Thus, we have the following theorem on the realization of the potential factor matrix of $(n+2)$ node-resistive n-port networks.

Theorem 3: Let a real matrix

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

be given. It can be realized as the potential factor matrix of an $(n+2)$ node-resistive n-port network containing no negative conductances and having a linear 2-tree port configuration defined by K_{11} and K_{22} if and only if the entries of the submatrices K_{12} and K_{21} of K satisfy the following conditions.

$$k_{ij} = k_{ik} = k_i \qquad i \le m$$

and

$$j, k = m + 1, m + 2, \cdots, n,$$

or

$$i \ge m + 1 \text{ and } j, k = 1, 2, \cdots, m,$$

and

$$1 > k_1 \ge k_2 \ge k_3 \ge \cdots \ge k_m$$

as the modified cut-set matrix of a resistive 4-port network having the port configuration shown in Fig. 2.

If C^* is realizable by a 4-port network N^* having the required port configuration, then the modified cut-set matrix C of the 4-port network N constructed on N^* and having the linear 2-tree port configuration shown in Fig. 3 can be obtained as $C = A'C^*$ where the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

relates the port voltage vector V_p^* of N^* to the port voltage vector V_p of N as $V_p^* = AV_p$.

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{7}{9} & \frac{7}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & \frac{5}{9} & \frac{5}{9} & \frac{5}{9} & \frac{5}{9} & -\frac{4}{9} & -\frac{4}{9} & -\frac{4}{9} & -\frac{4}{9} \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} \end{bmatrix}.$$

$$1 > k_{m+1} \ge k_{m+2} \ge k_{m+3} \ge \cdots \ge k_n.$$

Consider an n-port network N having a port configuration T that consists of two parts T_1 and T_2. Construct on the network N another n-port network N^* such that the port configuration T^* of N^* consists of two parts T_1^* and T_2^* with T_i^*, $i = 1, 2$ constructed on the same vertices as T_i. Then the potential factor matrix K^* of N^* can be obtained from the potential factor matrix K of N without making use of the conductance values of N. That is, the realizability of K^* can be tested by testing the realizability of K and vice versa. The process of transformation from K^* to K is not difficult. In view of these we can conclude that the necessary and sufficient conditions stated in Theorem 3 are general ones applicable to testing the realizability of the potential factor matrix of any $(n + 2)$ node-resistive n-port network.

We next illustrate the application of the above results in the realization of the modified cut-set matrix and Y matrix of $(n + 2)$ node-resistive n-port networks.

Example 1: Let it be required to realize the matrix

The potential factor matrix K of N obtained, using Theorem 2 of [2] is given by

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{7}{9} \\ 0 & 1 & 1 & \frac{5}{9} \\ 0 & 0 & 0 & \frac{3}{9} \\ \hline \frac{5}{9} & \frac{5}{9} & \frac{5}{9} & 1 \end{array}\right] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}.$$

From K, we get $K_1 = \frac{7}{9}$, $K_2 = \frac{5}{9}$, $K_3 = \frac{3}{9}$, and $K_4 = \frac{5}{9}$. It may be verified that the entries of K_{12} and K_{21} satisfy the conditions stated in Theorem 3. We next obtain a padding 4-port network having the above potential factor matrix K in which all $S_{1,2}$ are greater than zero.

Choose $S = 9$ and obtain $S_{1,2}$, $S_{2,2}$, $S_{3,2}$, and $S_{4,2}$ using (31) as

$$S_{1,2} = 2 \qquad S_{2,2} = 2 \qquad S_{3,2} = 2 \qquad S_{4,2} = 3.$$

Then using (37) and (39) the edge conductance matrix G_p of the required padding 4-port network N_p is obtained as

$$C^* = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & \frac{7}{9} & \frac{7}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & -\frac{7}{9} & -\frac{7}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} & -\frac{5}{9} & \frac{4}{9} \end{bmatrix}$$

$$G_p = \text{diag}\,\{g_{12}\quad g_{13}\quad g_{14}\quad g_{15}\quad g_{16}\quad g_{23}\quad g_{24}\quad g_{25}\quad g_{26}\quad g_{34}\quad g_{35}\quad g_{36}\quad g_{45}\quad g_{46}\quad g_{56}\}$$
$$= \text{diag}\,\{-\tfrac{4}{9}\quad -\tfrac{4}{9}\quad -\tfrac{6}{9}\quad \tfrac{8}{9}\quad \tfrac{10}{9}\quad -\tfrac{4}{9}\quad -\tfrac{6}{9}\quad \tfrac{8}{9}\quad \tfrac{10}{9}\quad -\tfrac{6}{9}\quad \tfrac{8}{9}\quad \tfrac{10}{9}\quad \tfrac{12}{9}\quad \tfrac{15}{9}\quad -\tfrac{20}{9}\}.$$

Next a suitable network of departure N_d is obtained, using the procedure stated in the proof of Property i), as

$$G_d = \text{diag}\,\{g_{12}\quad g_{13}\quad g_{14}\quad g_{15}\quad g_{16}\quad g_{23}\quad g_{24}\quad g_{25}\quad g_{26}\quad g_{34}\quad g_{35}\quad g_{36}\quad g_{45}\quad g_{46}\quad g_{56}\}$$
$$= \text{diag}\,\{\tfrac{4}{9}\quad \tfrac{4}{9}\quad \tfrac{6}{9}\quad 0\quad 0\quad \tfrac{4}{9}\quad \tfrac{6}{9}\quad 0\quad 0\quad \tfrac{6}{9}\quad 0\quad 0\quad 0\quad 0\quad \tfrac{20}{9}\}.$$

The 4-port network N obtained as the parallel combination of N_p and N_d and having the linear 2-tree port configuration shown in Fig. 3 will have C as its modified cut-set matrix and hence the 4-port network N^* constructed on N and having the port configuration shown in Fig. 2 will have the given C^* as its modified cut-set matrix. The 4-port network N^* is shown in Fig. 4.

Example 2: Let it be required to realize the matrix

$$Y = \begin{bmatrix} 61 & 45 & 16 & 5 \\ 45 & 95 & 40 & 5 \\ 16 & 40 & 56 & 0 \\ 5 & 5 & 0 & 35 \end{bmatrix}$$

as the short-circuit conductance matrix of a 4-port resistive network having a linear 2-tree port configuration shown in Fig. 5. The edge conductance matrix G_d of the network of departure with respect to Y and having the required port configuration is obtained as

$$G_d = \text{diag}\,\{g_{12}\quad g_{13}\quad g_{14}\quad g_{15}\quad g_{16}\quad g_{23}\quad g_{24}\quad g_{25}\quad g_{26}\quad g_{34}\quad g_{35}\quad g_{36}\quad g_{45}\quad g_{46}\quad g_{56}\}$$
$$= \text{diag}\,\{16\quad 29\quad 16\quad -5\quad 5\quad 26\quad 24\quad 0\quad 0\quad 16\quad 5\quad -5\quad 0\quad 0\quad 35\}.$$

Next we have to generate a 4-port padding network N_p having the specified port configuration and such that the parallel combination of N_p and N_d contains no negative conductances. Thus we obtain the following constraints on the edge conductances of N_p.

$$g_{15} = (1 - k_4)S_{1,2} \geq 5$$
$$g_{16} = K_4S_{1,2} \geq -5$$
$$g_{25} = (1 - K_4)S_{2,2} \geq 0$$
$$g_{26} = K_4S_{2,2} \geq 0$$
$$g_{35} = (1 - K_4)S_{3,2} \geq -5$$
$$g_{36} = K_4S_{3,2} \geq 5$$
$$g_{45} = (1 - k_4)S_{4,2} \geq 0$$
$$g_{46} = K_4S_{4,2} \geq 0$$
$$g_{12} = \frac{-S_{1,2}S_{2,2}}{S} \geq -16$$
$$g_{13} = \frac{-S_{1,2}S_{3,2}}{S} \geq -29$$
$$g_{56} = -(1 - K_4)K_4S \geq -35$$
$$g_{14} = \frac{-S_{1,2}S_{4,2}}{S} \geq -16$$
$$g_{23} = \frac{-S_{2,2}S_{3,2}}{S} \geq -26$$
$$g_{24} = \frac{-S_{2,2}S_{4,2}}{S} \geq -24$$
$$g_{34} = \frac{-S_{3,2}S_{4,2}}{S} \geq -16$$

where

$$S = (S_{1,2} + S_{2,2} + S_{3,2} + S_{4,2}).$$

A proper choice of K_4, $S_{1,2}$, $S_{2,2}$, $S_{3,2}$, and $S_{4,2}$ should be made so that the above constraints are satisfied. Further, the following conditions should be satisfied:

all $S_{i,2}$ should be greater than zero,
$1 > K_1 \geq K_2 \geq K_3$
$K_4 < 1$.

Two such choices are as follows:

$$K_4 = 5/10$$
$$S_{1,2} = 30 \qquad S_{2,2} = 30 \qquad S_{3,2} = 30 \qquad S_{4,2} = 20$$

and

$$K_4 = 5/10$$
$$S_{1,2} = 20 \qquad S_{2,2} = 10 \qquad S_{3,2} = 20 \qquad S_{4,2} = 10.$$

Using the above values suitable padding 4-port networks can be obtained. The final 4-port network corresponding to the first choice and realizing the given Y matrix is shown in Fig. 6.

V. Conclusions

In this paper we have considered the realization of the modified cut-set matrix of resistive n-port networks and certain aspects of its relation to the Y-matrix synthesis problem. Based on the properties of network of departure and padding n-port network, certain necessary and sufficient conditions are given for the realization of the modified cut-set matrix of n-port networks having a prescribed port configuration.

The new procedure suggested for the realization of the short-circuit conductance matrices of n-port networks having more than $(n + 1)$ nodes differs from that of

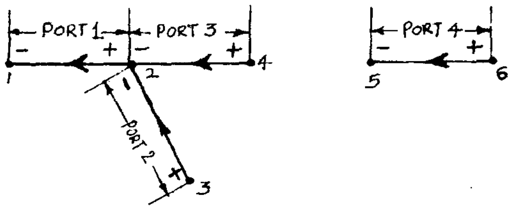

Fig. 2. Port configuration of network required in Example 1.

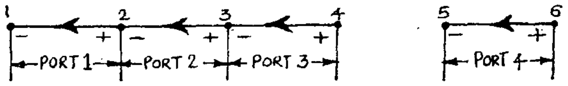

Fig. 3. Linear 2-tree port configuration used in Example 1.

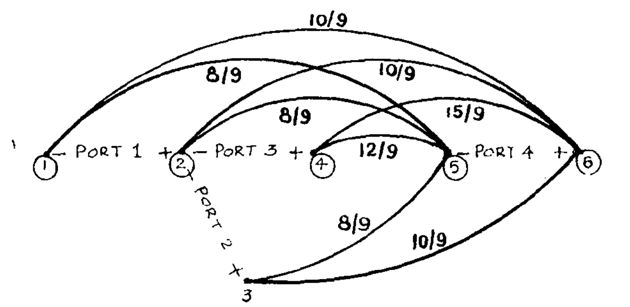

Fig. 4. Network realizing the modified cut-set matrix C^* of Example 1.

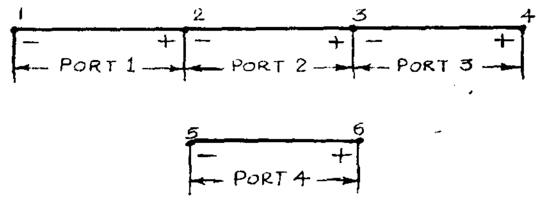

Fig. 5. Port configuration of network required in Example 2.

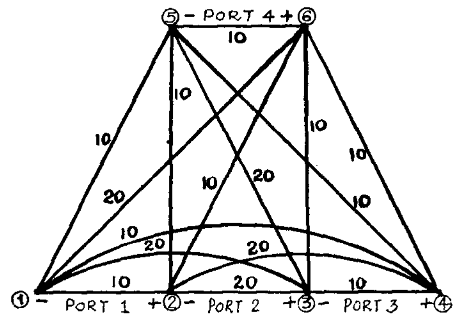

Fig. 6. Network realizing the Y matrix of Example 2.

Guillemin [4] in the method adopted to generate padding networks. The method used in this paper expresses edge conductances of padding networks in terms of potential factors. A first step towards systematizing the Y-matrix synthesis procedure suggested in this paper is to obtain the necessary and sufficient conditions that the potential factors of n-port network containing no negative conductances should satisfy. Such a condition has been obtained in this paper, for the special case of $(n + 2)$ node networks. Though a similar condition can be shown to be necessary in the general case, its sufficiency is yet to be established. In the meantime, the application of the results obtained to the synthesis of the Y matrix of an $(n + 2)$ node n-port network has been illustrated.

We wish to point out that the formulas expressing the edge conductances of $(n + 2)$ node-padding n-port networks, as obtained in this paper, can be given in terms of all potential factors and S, or all $S_{i,2}$ and $S_{i,1}$, or

all $S_{i,2}$ and all K_i $\quad i = m + 1, m + 2, \cdots, n$

or

all $S_{i,1}$ and all K_i $\quad i = 1, 2, \cdots, m.$

Such expressions will be useful, as stated in Section III, to extend these results to the synthesis of RLC n-port networks.

We note that the potential factors $S_{i,2}$ and $S_{i,1}$ and S are all the same both for the padding network and the final network that will be the parallel combination of the padding network and a suitable network of departure. Thus, the arbitrary quantities that are assumed in generating a padding network can be identified with the corresponding quantities in the final network required. The procedures used in [5] and [6] do not permit such straightforward identifications.

Another result of importance that requires special mention is that if a network is to be the padding network of an $(n + 2)$ node-resistive n-port network containing no negative conductances, then the conductances of the edges in this network connecting vertices in T_1 to vertices in T_2 should be nonnegative. This follows from the nonnegativeness of $S_{i,2}$ and $S_{i,1}$.

All the procedures presented so far (including the present one) for the synthesis of the Y matrix of resistive n-port networks [5], [6] essentially aim at generation of suitable padding networks. Therefore, it is quite possible to start from one set of formulas for edge conductances of padding networks and obtain another set.

Even though some of the procedures given in this paper require further study and systematization, it is hoped that the theory developed in this paper provides more insight into several aspects of analysis and synthesis of n-port networks. The authors have obtained certain necessary conditions and certain sufficient conditions for the synthesis of the Y-matrix $(n + 2)$ node-resistive n-port networks. Further, the application of the techniques discussed here has led to some results on the lower bound on the number of conductances and amount of conductance required for the realization of a real symmetric matrix as the short-circuit conductance matrix of n-port networks having more than $(n + 1)$ nodes.

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Connectivity Considerations in the Design of Survivable Networks

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***Abstract*—The problem of constructing networks that are "survivable" with respect to branch damage is considered. The networks are modeled by linear graphs and a square symmetric "redundancy" matrix $R' = [r'_{ij}]$ is specified. Algorithms are given to construct an undirected graph G with a minimum number of branches such that 1) G contains no parallel branches, and 2) for all i, j there are at least r'_{ij} branch disjoint paths between the ith and jth vertices. These algorithms are complicated but may easily be applied to construct graphs with several hundred vertices.**

Introduction

A PRIME consideration in the design of many networks is that the network "survive" an enemy attack. The desired type of survivability depends on the nature of the threat and the function of the network. In a number of cases, a reasonable measure of survivability is the number of links and/or stations that must be destroyed before communication is disrupted.

Networks may be modeled by linear graphs so that stations correspond to vertices and links correspond to branches. With this model, a number of authors have studied various aspects of analysis and synthesis with survivability as a parameter [1]–[10]. Some of the synthesis studies have been concerned with maximizing the minimum number of vertices and/or branches that must be removed from the graph before the graph becomes disconnected [1]–[5]. Steiglitz *et al.* [6] have considered the general minimum-cost synthesis problem and have obtained many important results using heuristic programming. Other studies have concentrated on developing efficient methods of finding the minimum numbers of vertices and/or branches that must be removed to separate specified pairs of vertices.

Manuscript received July 25, 1968; revised March 10, 1970. This research was supported by the Executive Office of the President, Office of Emergency Preparedness, Washington, D. C., while H. Frank was on leave from the University of California, Berkeley, and by the Network Analysis Corporation, Glen Cove, N. Y.

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In this paper, the problem of constructing networks survivable with respect to branch damage is examined. Such damage could be of particular concern in networks subject to sabotage or other threats to which stations are relatively invulnerable. Specifically, suppose an $n' \times n'$ symmetric matrix $R' = [r'_{ij}]$ of nonnegative integers is specified such that $r'_{ii} = 0$ for $i = 1, \cdots, n'$. The problem considered is: construct an undirected graph G without parallel branches and with fewest branches so that there are at least r'_{ij} branch disjoint paths between the ith and jth vertices for $i, j = 1, 2, \cdots, n'$. In the communication network corresponding to such a graph, at least r'_{ij} branches must be destroyed before communication between the ith and jth stations is totally disrupted.

Preliminary Considerations

Let G be an undirected graph without parallel branches, defined by the ordered pair (V, Γ) (written $G = (V, \Gamma)$). $V = \{v_1, \cdots, v_n\}$ is the set of vertices of G, and Γ is the set of branches defined by $\Gamma = \{[v_i, v_j]\}$ such that $[v_s, v_t] \in \Gamma$ if and only if there is a branch between v_s and v_t. Branch $[v_s, v_t] \in \Gamma$ is said to be *incident* at vertices v_s and v_t, and v_s and v_t are said to be *adjacent*.

Associated with $[v_i, v_j]$ is a capacity $c[v_i, v_j]$ representing the total amount of allowable flow in $[v_i, v_j]$. The maximum possible flow between v_i and v_j is known as the *terminal capacity* t_{ij} and can be determined by the max-flow min-cut theorem [11]. The $n \times n$ matrix $T = [t_{ij}]$ where $t_{ii} = 0$ for $i = 1, \cdots, n$ is called the *terminal capacity matrix* [12]. Since G is undirected, T is symmetric.

The degree $d(i)$ of a vertex v_i is the number of branches incident at that vertex. Given G, it is easy to find vertex degrees. A less trivial problem is: given a set of positive integers $k_1, k_2, \cdots, k_n$, construct an n-vertex graph G without parallel branches whose vertices have degrees $k_1, \cdots, k_n$. Not all sets of positive integers can be realized